Representations of $\mathrm{SL}_2(\mathbb{K})$ as $\mathrm{SL}_2(\mathbb{K})$-modules

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$\mathrm{SL}_2(\mathbb{K})$

The talk is about the group $\mathrm{SL}_2(\mathbb{K})$.

Definition

For $\mathbb{K}$ a field, we let:

$$\mathrm{SL}_2(\mathbb{K}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : (a, b, c, d) \in \mathbb{K}^4 : ad - bc = 1 \right\}$$

be the group of 2×2 matrices with determinant 1.

We are interested in some representations of this group.

What is a group?

Be careful that $\mathrm{SL}_2(\mathbb{K})$ can be considered:

- as a Lie group if $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$;
- as an algebraic group (the group of $\mathbb{K}$-points of an alg. group);
- as a finite group (if $\mathbb{K}$ is finite);
- as an abstract group (= a group, and nothing more).

Accordingly, there can be more than the sole group structure.
In particular, the notion of a representation depends on how we consider $\mathrm{SL}_2(\mathbb{K})$ ("in which category?").
In this talk we go for less: $\mathrm{SL}_2(\mathbb{K})$ will be an abstract group.

Homogeneous polynomial representations

Let (X, Y) be indeterminates. $\mathrm{SL}_2(\mathbb{K})$ acts from the right by:

$$(X, Y) \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (aX + cY, bX + dY)$$

Now let $\mathrm{SL}_2(\mathbb{K})$ act on polynomials $P(X, Y)$ from the left by:

$$\underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{g} \cdot P(X, Y) = P((X, Y) \cdot g) = P(aX + cY, bX + dY)$$

This does define an action. Degree is preserved, so we restrict:

Fact

For each $d \geq 1$, $\mathrm{SL}_2(\mathbb{K})$ acts on the $\mathbb{K}$-vector space of homogeneous polynomials of degree d in (X, Y).

Homogeneous polynomial representations, cont'd

Notation

Let $S_d(\mathbb{K})$ be the space of homogeneous polynomials of degree d in (X, Y), equipped with the action of $\mathrm{SL}_2(\mathbb{K})$.

So $S_d(\mathbb{K}) = \mathbb{K}[X^d, X^{d-1}Y, \ldots, XY^{d-1}, Y^d]$ with : $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot P(X, Y) = P(aX + cY, bX + dY)$

Fact

$S_d(\mathbb{K})$ has dimension $d + 1$ and is irreducible, i.e. there are no $\mathrm{SL}_2(\mathbb{K})$-invariant vector subspaces (other than 0 and $S_d(\mathbb{K})$).

Example

- The natural representation on $\mathbb{K}^2$ can be viewed as S_1.
- The adjoint representation can be viewed as S_2.

  Adjoint rep.: action by conjugacy on 2×2 matrices with trace 0.

Representations of $\mathrm{SL}_2(\mathbb{K})$ as an algebraic group

Fact

$S_d(\mathbb{K}) = \mathbb{K}[X^d, X^{d-1}Y, \ldots, XY^{d-1}, Y^d]$ *is an irreducible rep. of* $\mathrm{SL}_2(\mathbb{K})$.

($S_d \simeq \mathrm{Sym}^d \,\mathrm{Nat}\,\mathrm{SL}_2$, but we won't use that.)

Fact (algebraic representation theory)

The irreducible representations of the alg. group SL_2 *are exactly the spaces* S_d*, for* $d \geq 1$.

In the algebraic category, this is the end of the story.

Question

But as an *abstract* group?

The talk aims at addressing this question.

The question

Question

Can one try to understand abstract $\mathrm{SL}_2(\mathbb{K})$-modules?
Can one identify $S_d(\mathbb{K})$ among $\mathrm{SL}_2(\mathbb{K})$-modules?

Definition

A G-module is an abelian group V acted on by G.
We do not assume that V is a vector space.

No extra structure on V nor G...
So let's do brute-force group theory.

Known cases

"Can one identify $S_d(\mathbb{K})$ among $\mathrm{SL}_2(\mathbb{K})$-modules?"

Theorem (Smith, Timmesfeld "Quadratic" Theorem)

Let V be an irreducible $\mathrm{SL}_2(\mathbb{K})$-module. Suppose $(u-1)^2 = 0$ in $\mathrm{End}(V)$, where $u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.
Then $V \simeq S_1(\mathbb{K}) = \mathbb{K}[X, Y]$ (natural rep.).

The proof gives a way to explicitly *define* a $\mathbb{K}$-linear structure on V.

Theorem (Cherlin-D.)

Let V be an irreducible $\mathrm{SL}_2(\mathbb{K})$-module with Morley rank $2\,\mathrm{rk}\,\mathbb{K} < \mathrm{rk}\,V \leq 3\,\mathrm{rk}\,\mathbb{K}$.
Then $V \simeq S_2(\mathbb{K}) = \mathbb{K}[X^2, XY, Y^2]$ (adjoint rep.).

The latter served as the starting point of the talk.

Strategy

Goal: identify $S_d(\mathbb{K})$ among abstract $\mathrm{SL}_2(\mathbb{K})$-modules.

Notation

Let $\mathbb{K}$ be a field and $\mathbb{K}_1 \subseteq \mathbb{K}$ be the prime subfield ($\mathbb{F}_p$ or $\mathbb{Q}$).
Let $G = \mathrm{SL}_2(\mathbb{K})$ and $G_1 = \mathrm{SL}_2(\mathbb{K}_1) \leq \mathrm{SL}_2(\mathbb{K}) = G$.

Ideology

$G = \mathrm{SL}_2(\mathbb{K})$ is *flesh and bones*: it has

- a skeleton, visible in $G_1 = \mathrm{SL}_2(\mathbb{K}_1)$;
- flesh on it, extending $\mathrm{SL}_2(\mathbb{K}_1)$ into $\mathrm{SL}_2(\mathbb{K})$.

Strategy (two-fold strategy)

1. Understand $\mathrm{SL}_2(\mathbb{K}_1)$-modules.
2. See if $\mathrm{SL}_2(\mathbb{K}_1)$-structure of module determines $\mathrm{SL}_2(\mathbb{K})$-structure.

Scalar extension

We begin with Task 2:

2. See if $\mathrm{SL}_2(\mathbb{K}_1)$-structure of module determines $\mathrm{SL}_2(\mathbb{K})$-structure.

We need to understand how $G = \mathrm{SL}_2(\mathbb{K})$ extends $G_1 = \mathrm{SL}_2(\mathbb{K}_1)$, and how the G_1-structure constraints the G-structure.
This is *not* "Clifford theory" since G_1 is highly non-normal in G!

A result

$V \simeq_H W$ means: V and W are isomorphic *as H-modules*.

Theorem (The Flesh Theorem)

Let V be a $G = \mathrm{SL}_2(\mathbb{C})$-module and $d \geq 1$ be fixed. Suppose that over $G_1 = \mathrm{SL}_2(\mathbb{Q})$, $V \simeq_{G_1}$ a sum of copies of $S_d(\mathbb{Q})$.
Then $V \simeq_G$ a sum of copies of $S_d(\mathbb{C})$.

Remark

The theorem holds for arbitrary $\mathbb{K}$ provided the characteristic is large enough ($\sim 2n$) and $\mathbb{K}$ has enough roots ($\sim n!$-rad. closed).

Remark

Same theorem for $\mathfrak{sl}_2(\mathbb{K})$ (no assumptions on $\mathbb{K}$ being rad. closed).

Two words on the proof

Theorem

Suppose $V \simeq_{G_1}$ a sum of copies of $S_d(\mathbb{Q})$ (fixed d). Then $V \simeq_G$ a sum of copies of $S_d(\mathbb{C})$.

- One must *construct* a $\mathbb{C}$-vector space structure on V.
- Idea: use unipotent group to capture weight spaces, then use torus to define scalar action.
- Problem: torus $T \simeq \mathbb{K}^\times$ gives multiplication but no addition! $(\lambda\mu) \cdot v = t_{\lambda\mu} v = t_\lambda t_\mu v = \lambda \cdot (\mu \cdot v)$, but $(\lambda + \mu) \cdot v = t_{\lambda+\mu} \cdot v = ?$

Proof sketch (for experts)

Let $u_\lambda = \begin{pmatrix} 1 & \lambda \\ & 1 \end{pmatrix}$ and $t_\lambda = \begin{pmatrix} \lambda & \\ & \lambda^{-1} \end{pmatrix}$. Also let $w = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$.
In $\mathrm{End}(V)$, let $x_\lambda = \log(u_\lambda)$.

- Prove that x_λ makes sense.
- Capture weight spaces in terms of x_1 (visible over G_1!)
- For $a \in C_V(u_1) = \ker x_1 = \ker x_\lambda = C_V(u_\lambda)$, see that:

$$x_1^{d-1} x_{\lambda^d} w \cdot a = t_\lambda \cdot a$$

  (Use "Steinberg relations".)
- This proves that letting $\lambda^d \cdot a := t_\lambda \cdot a$ is well-defined and additive in λ!
- Push linear structure around with w and x_1; get linearity. □

A question

Theorem

Suppose $V \simeq_{G_1}$ a sum of copies of $S_d(\mathbb{Q})$ (fixed d). Then $V \simeq_G$ a sum of copies of $S_d(\mathbb{C})$.

More generally:

Question

Let $\mathbb{G}$ be an algebraic group and $\mathbb{K} \subseteq \mathbb{L}$ be two fields.
Let V be a $\mathbb{G}_\mathbb{L}$-module. Suppose V is isotypical as a $\mathbb{G}_\mathbb{K}$-module.
Can one describe the $\mathbb{G}_\mathbb{L}$-structure?

Non-experts, please wake up! Now for some colourful geometry.

The goal

We return to Task 1:

1. Understand $\mathrm{SL}_2(\mathbb{K}_1)$-modules.

Here we are too ambitious: we shall even work with $G_0 = \mathrm{SL}_2(\mathbb{Z})$.
G_0 has no torus, and no Bruhat decomposition $\rightarrow$ much trickier.

Remark

In S_d, element $u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ acts with $(u-1)^{d+1} = 0$.

This suggests to focus on modules with $(u-1)^k = 0$.

A result

Theorem (The Bones Theorem)

Let V be a $\mathbb{Q}[\mathrm{SL}_2(\mathbb{Z})]$-module. Suppose $(u-1)^5 = 0$ in $\mathrm{End}(V)$. Then V has a composition series $0 = V_0 \leq \cdots \leq V_i \leq \cdots \leq V_n$ s.t. every factor is a sum of $S_{d_i}(\mathbb{Q})$ (d_i depends only on i).

Proof.

Ugly computation involving $\tanh(\frac{1}{2}\log u)$ $\leftarrow$ (hyperbolic stuff!). □

Corollary (combine with "Flesh" Theorem)

1. *Smith-Timmesfeld: if $(u-1)^2 = 0$ then V is a sum of $S_1(\mathbb{K})$.*
2. *V irreducible with $2\,\mathrm{rk}\,\mathbb{K} < \mathrm{rk}\,V \leq 3\,\mathrm{rk}\,\mathbb{K} \Rightarrow V \simeq S_2(\mathbb{K})$.*

Questions

Theorem (The Bones Theorem)

Let V be a $\mathbb{Q}[\mathrm{SL}_2(\mathbb{Z})]$-module. Suppose $(u-1)^5 = 0$ in $\mathrm{End}(V)$. Then V has a composition series $0 = V_0 \leq \cdots \leq V_i \leq \cdots \leq V_n$ s.t. every factor is a sum of $S_{d_i}(\mathbb{Q})$ (d_i depends only on i).

But Step 1 of strategy was for $\mathrm{SL}_2(\mathbb{K}_1)$, $\mathbb{K}_1$ the *prime field*!

Questions

- What happens with $\mathbb{Q}$ instead of $\mathbb{Z}$?
- What happens with $\mathbb{F}_p$ instead of $\mathbb{Z}$ and $\mathbb{Q}$?

I do not know.

A digression: the Lie ring

Theorem (The Bones Theorem)

Let V be a $\mathbb{Q}[\mathrm{SL}_2(\mathbb{Z})]$-module. Suppose $(u-1)^5 = 0$ in $\mathrm{End}(V)$. Then V has a composition series $0 = V_0 \leq \cdots \leq V_i \leq \cdots \leq V_n$ s.t. every factor is a sum of $S_{d_i}(\mathbb{Q})$ (d_i depends only on i).

Let $\mathfrak{sl}_2(\mathbb{K}) = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} : (a, b, c) \in \mathbb{K}^3 \right\}$ seen as *Lie ring*, i.e. with $+$ and $[\cdot, \cdot]$ (forget vector space structure).

Theorem

Let V be a $\mathbb{K}_1[\mathfrak{sl}_2(\mathbb{K}_1)]$-module, $\mathbb{K}_1 = \mathbb{Q}$ or $\mathbb{F}_p$ with $p > 2n$.

Suppose $x^n = 0$ in $\mathrm{End}(V)$, where $x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

Then V splits into a direct sum of $S_d(\mathbb{K}_1)$ (various d's).

A corollary

Theorem (The Bones Theorem)

Let V be a $\mathbb{Q}[\mathrm{SL}_2(\mathbb{Z})]$-module. Suppose $(u-1)^5 = 0$ in $\mathrm{End}(V)$. Then V has a composition series $0 = V_0 \leq \cdots \leq V_i \leq \cdots \leq V_n$ s.t. every factor is a sum of $S_{d_i}(\mathbb{Q})$ (d_i depends only on i).

Corollary

The $\mathbb{Q}$-algebra $\mathbb{Q}[\mathrm{SL}_2(\mathbb{Z})]/((u-1)^5)$ (quotient by an ideal) is finite-dimensional.

Limits

Corollary

$\mathbb{Q}[\mathrm{SL}_2(\mathbb{Z})]/((u-1)^5)$ *is finite-dimensional.*

Proposition (Wolff)

$\mathbb{Q}[\mathrm{SL}_2(\mathbb{Z})]/((u-1)^7)$ is not finite-dimensional.

Proof.

Retrieve triangular group $D(2,3,7)$. $\longleftarrow$ hyperbolic stuff again!! $\square$

Question

What about $n = 6$?

Geometry

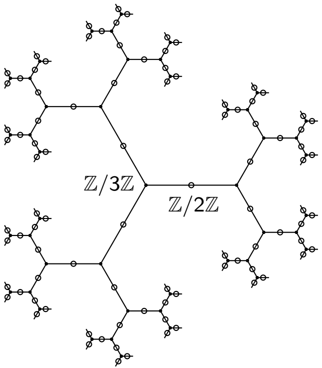
Corollary

$\mathbb{Q}[\mathrm{SL}_2(\mathbb{Z})]/((u-1)^5)$ *is finite-dimensional.*

I wish to give an alternate, geometric proof of the Corollary.
The argument was entirely found by Maxime Wolff.
It is highly visual and involves Bass-Serre theory.
Even if you don't get all details, it will be fun.
(And if you're not a geometer, you might learn something.)

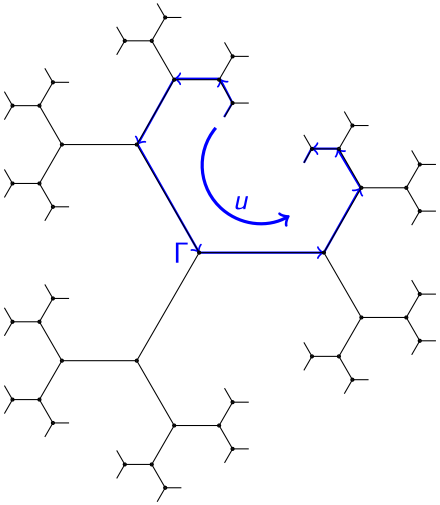
The Tree

Bass-Serre tree of $\mathrm{PSL}_2(\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$ (combinatorial object):



Arity 2 nodes bear no information: forget them.

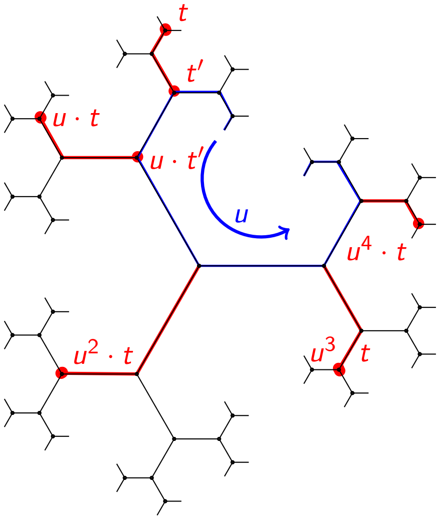
The Action



$G_0 = \mathrm{SL}_2(\mathbb{Z})$ acts on T (set of vertices).

- A *special geodesic* is a line that always turns in the same direction.
- Translation along Γ makes sense.
- Fact: each conjugate of $u^{\pm 1} = \begin{pmatrix} 1 & \pm 1 \\ & 1 \end{pmatrix}$ acts as a 1-step translation along some special geodesic.

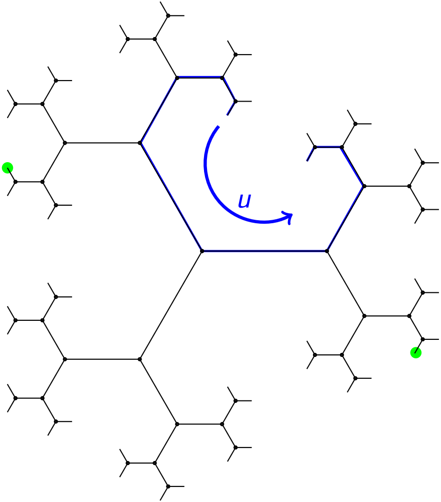
An Example



Suppose we want the iterates $u^k \cdot t$ of given t ($k \geq 0$).

- Link t to the axis of u – call t' the projection.
- Compute $u \cdot t'$ (easy).
- Then copy-paste the red segment, moving it along the axis.
- And so on.

Another Example: find $u^2 \cdot t$



The Proof – Changing model

Goal (the corollary)

$\mathbb{Q}[\mathrm{SL}_2(\mathbb{Z})]/((u-1)^n)$ is finite-dimensional for $n \leq 5$. We'll do it for $n = 4$.

- Let $Y = \{gu^{\pm 1}g^{-1} : g \in \mathrm{SL}_2(\mathbb{Z})\} \leftrightarrow \{$1-step spec. geod. transl.$\}$.
- Let T be the set of vertices and $M = \mathbb{Q}[T]$ (v.s. with basis T).

  Action of $\mathrm{SL}_2(\mathbb{Z})$ on T extends linearly to an action on M.
- Let $N = \langle\{(u_1 - 1)^n \cdot t : (u_1, t) \in Y \times T\}\rangle$, subspace of M.
- Now let $Q = M/N$.

Fact (mostly changing language)

$\mathbb{Q}[\mathrm{SL}_2(\mathbb{Z})]/((u-1)^n)$ *is finite-dimensional iff Q is.*

Goal (redefined)

$Q = M/N$ is finite-dimensional for $n \leq 5$.

The Proof – Changing point of view

Notation

$Y = \{gu^{\pm 1}g^{-1} : g \in \mathrm{SL}_2(\mathbb{Z})\}$ $\quad M = \mathbb{Q}[T]$
$N = \langle\{(u_1 - 1)^n \cdot t : (u_1, t) \in Y \times T\}\rangle$ $\quad Q = M/N$

Goal (redefined)

- $Q = M/N$ is finite-dimensional for $n \leq 5$ (we'll do it for $n = 4$).
- M/N has a basis in a finite set around some fixed $t_0 \in T$.
- Any $t \in T$ far from t_0 is lin. comb. of elmnts *closer to* t_0.

Now observe that for arbitrary $(t, u_1) \in T \times Y$,

$$(u_1 - 1)^n \cdot t = u_1^n \cdot -nu_1^{n-1} \cdot t + \cdots + (-1)^{n-1}nu_1 \cdot t + (-1)^n t$$

so (in Q), $t =$ a linear combination of $u_1 \cdot t, \ldots, u_1^n \cdot t$.

Goal (redefined)

Push far away t closer to fixed t_0 using spec. geodesic translations.

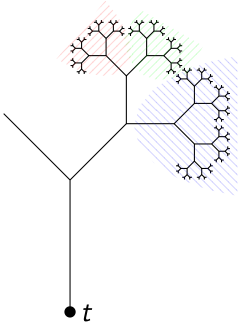
The Proof – Three Cases

Goal (redefined)

Push far away t closer to fixed t_0 using spec. geodesic translations.

(It's OK if you missed the point: one may start from here.)

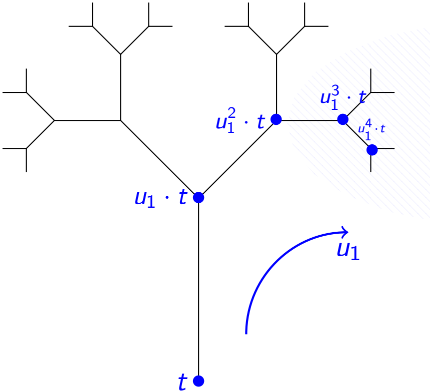
There are three cases, depending on where the origin t_0 is.



We are trying to push t closer to t_0 but:

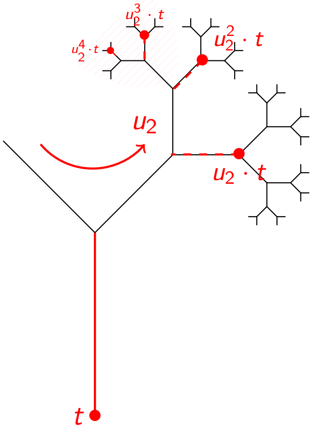
- point marked is t,
- case div. is on t_0.

The Proof – Case 1



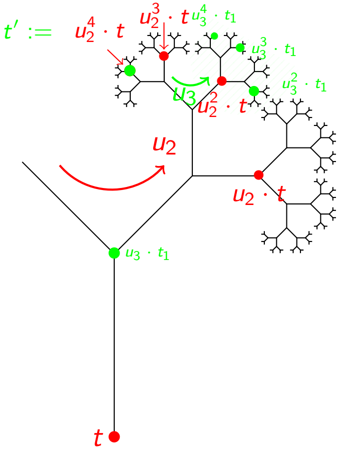
Suppose t_0 lies in the blue zone. Let us push t and iterates closer.
$u_1 \cdot t$, $u_1^2 \cdot t$, $u_1^3 \cdot t$ and $u_1^4 \cdot t$ are closer to t_0 than t was: done!

The Proof – Case 2



If t_0 lies in red zone, u_2 takes $u_2 \cdot t, \ldots, u_2^4 \cdot t$ closer: done!

The Proof – Case 3



Suppose t_0 lies in the green zone.

- Start like in Case 2.
- $u_2 \cdot t, u_2^2 \cdot t, u_2^3 \cdot t$ are closer to t_0...

  but $u_2^4 \cdot t$ is equally far as t (4 steps from green zone)!
- So it suffices to push $t' := u_2^4 \cdot t$ closer to t_0, with another unipotent.
- Pick u_3 and check.
- Done!

For Students

Exercises

1. Carry a proof with $n = 5$.
2. [not too hard] Translate your geometric proof into a formal argument.
3. [much harder] Translate your geometric proof into a beamer/tikz show-and-tell.
4. [open] What happens with $n = 6$?

Flesh and bones: Sum-up

Theorem (The Flesh Theorem)

Let V be a $G = \mathrm{SL}_2(\mathbb{C})$-module and $d \geq 1$ be fixed. Suppose that over $G_1 = \mathrm{SL}_2(\mathbb{Q})$, $V \simeq_{G_1}$ a sum of copies of $S_d(\mathbb{Q})$.
Then $V \simeq_G$ a sum of copies of $S_d(\mathbb{C})$.

Theorem (The Bones Theorem)

Let V be a $\mathbb{Q}[\mathrm{SL}_2(\mathbb{Z})]$-module. Suppose $(u-1)^5 = 0$ in $\mathrm{End}(V)$.
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Proposition (Wolff)

$\mathbb{Q}[\mathrm{SL}_2(\mathbb{Z})]/((u-1)^7)$ is not finite-dimensional.

Thanks!